The SU(2)(+)h(4) Hamiltonian

1990 J. Phys. A: Math. Gen. 23 75

(http://iopscience.iop.org/0305-4470/23/1/014)

J. Phys. A: Math. Gen. 23 (1990) 75-85. Printed in the UK

The SU(2)⊕h(4) Hamiltonian

C M Cheng and P C W Fung
Physics Department, University of Hong Kong, Pokfulam Road, Hong Kong

Received 23 May 1989

Abstract. In a previous paper we have employed a group theoretic method to find the evolution operator for a quantum system having a SU(2) Hamiltonian. In this paper, we consider a more complicated system whose Hamiltonian consists of SU(2) and h(4) group generators. A transformation method will be introduced, in conjunction with the above group theoretic method, to tackle this quantum problem. The result thus obtained will be applied to the problem of a mass-varying harmonic oscillator under an external force. Our result shows that the equation of motion of this oscillator is identical to a damped harmonic oscillator under an external force. In addition, it is shown that an initial coherent state will evolve as a squeezed state under the above Hamiltonian.

1. Introduction

In a previous paper [1], we have discussed the application of group theoretic methods [1-4] in solving the Schrödinger equation. It has been demonstrated that the evolution operator can be evaluated easily once the group property of the Hamiltonian of a quantum system is taken into account. More precisely, if the Hamiltonian consists of group generators of a closed Lie group, then the corresponding evolution operator can be represented by a product of exponential operators whose exponents contain the closed Lie group generators. In the previous paper, we considered a quantum system whose Hamiltonian consists of SU(2) group generators. It is shown that under such a Hamiltonian, a coherent state will evolve as a squeezed coherent state. This result has been applied to the consideration of a mass-varying harmonic oscillator [1, 5-13]. We have shown that this type of quantum harmonic oscillator is intimately related to a classical damped harmonic oscillator.

In order to explore the applicability of the group theoretic method, we shall, in this paper, consider a more complicated quantum system. The Hamiltonian of this system contains, besides the SU(2) group generators, the additional h(4) group (Heisenberg-Weyl) generators. This Hamiltonian can be applied to the analysis of the generation of non-Poissonian effects in laser-plasma scattering [14] and in the problem of a mass-varying harmonic oscillator under an external force [15-17]. Our aim here is to analyse how the evolution property of a coherent state is modified with the introduction of the h(4) group generators. In addition, we want to illustrate how we can modify the original method presented in our previous paper in order to tackle a more complicated Hamiltonian system.

In principle, we can utilise the method presented in the previous paper to treat the above complicated Hamiltonian system. However, due to the complex group structure of this Hamiltonian, it is not convenient and effective to evaluate the evolution operator for this Hamiltonian by means of the method in the previous paper. As a matter of

0305-4470/90/010075+11$03.50

fact, in the analysis of a *Hermitian* Hamiltonian possessing a SU(1, 1)⊕h(4) group structure, Dattoli *et al* [18] have utilised the notion of the interaction picture, in conjunction with a transformation method, to simplify the group structure of their Hermitian Hamiltonian. This procedure enables them to find the expression for the evolution operator. In our problem, however, the Hamiltonian need not be Hermitian. Therefore we shall introduce another transformation method to tackle our problem. This method has the advantage of being simple, straightforward and general. Besides, this transformation method allows us to utilise most of the results appearing in the previous paper. Therefore, much less effort is needed in the derivation of the evolution operator for our present problem.

The arrangement of this paper is as follows. In section 2, we shall develop the above-mentioned transformation method for a Hamiltonian system underlying a SU(2)⊕h(4) group structure. The evolution operator for such a system will be evaluated in exact form.

In section 3, we shall apply the method of section 2 to the problem of a mass-varying harmonic oscillator under an external force. This problem has been considered by Tartaglia [15] in the discussion of the motion of a particle in a viscous medium. Dodonov and Man'ko [16] have also discussed the evolution of a coherent state in such a quantum system. In addition, Khandekan and Lawande [17] have employed Feynman's propagator to solve this problem. We shall in this section discuss the effect of a sinusoidal external force on the oscillator. The evolution operator for such a system is found explicitly. The wavefunction is subsequently derived and is used to discuss the evolution property of an initial coherent state. Furthermore, the expectation values for the energy, position and momentum are evaluated. These results clearly elucidate the effect of an external force on a mass-varying oscillator.

Section 4 is devoted to our conclusion.

2. Evolution operator

Before we derive the expression for the evolution operator, let us recall the group theoretic method employed in the previous paper. If a Hamiltonian can be expressed as a linear combination of group generators of certain closed Lie algebra $\mathfrak{L}$,

$$\hat{H}(t)=\sum_{i=1}^{m} a_i(t)\hat{H}_i \tag{2.1}$$

where $a_i(t)$ are functions of time and $\{\hat{H}_i, i=1,\ldots,m\}$ form a closed Lie algebra $\mathfrak{L}$ of dimension m, then the corresponding evolution operator can be expressed locally in the following form:

$$\hat{U}(t)=\prod_{i=1}^{m} \exp[c_i(t)\hat{H}_i] \tag{2.2}$$

in which $c_i(t)$ are functions of time. In this way, the evaluation of the evolution operator $\hat{U}(t)$ is reduced to the determination of the functions $c_i(t)$.

In this paper we consider the Hamiltonian which contains a linear combination of SU(2)⊕h(4) group generators:

$$\hat{H}(t)=a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_- + a_4(t)\hat{K}_+ + a_5(t)\hat{K}_- + a_6(t) \tag{2.3}$$

where $a_i(t)$ are functions of time. In the above expression, the operators $\hat{J}_+$, $\hat{J}_-$ and $\hat{J}_0$ form an SU(2) Lie algebra

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_0 \tag{2.4}$$

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm \tag{2.5}$$

while the other three operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{1}$ form the Heisenberg-Weyl algebra, h(4) Lie algebra

$$[\hat{K}_+, \hat{K}_-] = -\hat{1}. \tag{2.6}$$

The commutation relations among the two sets of operators are given by

$$[\hat{J}_\pm, \hat{K}_\pm] = 0 \tag{2.7}$$

$$[\hat{J}_\pm, \hat{K}_\mp] = -\hat{K}_\pm \tag{2.8}$$

$$[\hat{J}_0, \hat{K}_\pm] = \pm\tfrac{1}{2}\hat{K}_\pm. \tag{2.9}$$

A straightforward approach to the above quantum system will be the direct application of the result presented by (2.2) to the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle = \hat{H}(t)|\Phi(t)\rangle \tag{2.10}$$

so that a set of six differential equations can be deduced. This set of equations can subsequently be solved to give the coefficients $c_i(t)$ of the evolution operator (cf (2.2)). However the setting up of the six equations is rather tedious. We therefore resort to another approach based on a transformation method.

To start with, we invoke a transformation on the Schrödinger wavefunction $|\Phi(t)\rangle$ represented by

$$|\Phi(t)\rangle = \hat{U}_1(t)|\Psi(t)\rangle \tag{2.11}$$

where $|\Psi(t)\rangle$ represents a transformed wavefunction, while $\hat{U}_1(t)$ is a unitary operator given by

$$\hat{U}_1(t) = \mathrm{e}^{c_1(t)\hat{J}_+}\,\mathrm{e}^{c_2(t)\hat{J}_0}\,\mathrm{e}^{c_3(t)\hat{J}_-}. \tag{2.12}$$

In order that the unitary operator satisfies the initial condition

$$\hat{U}_1(0) = \hat{I} \tag{2.13}$$

the coefficients $c_i(t)$ are required to obey the following conditions:

$$c_i(0) = 0 \qquad i = 1, 2, 3. \tag{2.14}$$

The explicit form of the coefficients $c_i(t)$ is not yet specified. Resetting the transformation (2.11) into (2.10), we arrive at the following equation in the $|\Psi(t)\rangle$ representation:

$$\hat{U}_1^+(t)\left(\hat{H}(t) - \mathrm{i}\hbar\frac{\partial}{\partial t}\right)\hat{U}_1(t)|\Psi(t)\rangle = 0. \tag{2.15}$$

Now the coefficients $c_i(t)$ appearing in $\hat{U}_1(t)$ are specified by requiring that only the expression

$$\hat{U}_1^+(t)\hat{H}(t)\hat{U}_1(t) - \mathrm{i}\hbar\hat{U}_1^+(t)\left(\frac{\partial}{\partial t}\hat{U}_1(t)\right) \tag{2.16}$$

be independent of the operators $\hat{J}_+$, $\hat{J}_-$ and $\hat{J}_0$. In other words, we require the expression (2.16) to be comprising only the operators $\hat{K}_+$, $\hat{K}_-$ and a constant term. By inserting (2.3) and (2.12) into the expression (2.16), we obtain

$$\hat{U}_1^+(t)\hat{H}(t)\hat{U}_1(t) - \mathrm{i}\hbar\hat{U}_1^+(t)\left(\frac{\partial}{\partial t}\hat{U}_1(t)\right)$$
$$\equiv \zeta_1(t)\hat{J}_+ + \zeta_2(t)\hat{J}_0 + \zeta_3(t)\hat{J}_- + \zeta_4(t)\hat{K}_+ + \zeta_5(5)\hat{K}_- + \zeta_6 \tag{2.17}$$

where the $\zeta_i(t)$ are given by

$$\zeta_1 \equiv e^{-c_2}[-\mathrm{i}\hbar\dot{c}_1 + a_1 + a_2c_1 - a_3c_1^2] \tag{2.18}$$

$$\zeta_2 \equiv -\mathrm{i}\hbar[\dot{c}_2 + 2\dot{c}_1c_3\,\mathrm{e}^{-c_2}] + 2a_1c_3\,\mathrm{e}^{-c_2} + a_2[1 + 2c_1c_3\,\mathrm{e}^{-c_2}] - 2a_3c_1[1 + c_1c_3\,\mathrm{e}^{-c_2}], \tag{2.19}$$

$$\zeta_3 \equiv -\mathrm{i}\hbar[\dot{c}_3 - c_3\dot{c}_2 - \dot{c}_1c_3^2\,\mathrm{e}^{-c_2}] - a_1c_3^2\,\mathrm{e}^{-c_2} - a_2c_3[1 + c_1c_3\,\mathrm{e}^{-c_2}]$$
$$+ a_3[\mathrm{e}^{c_2} + 2c_1c_3 + c_1^2c_3^2\,\mathrm{e}^{-c_2}] \tag{2.20}$$

$$\zeta_4 \equiv [a_4 + a_5c_1]\,\mathrm{e}^{-(1/2)c_2} \tag{2.21}$$

$$\zeta_5 \equiv a_4c_3\,\mathrm{e}^{-(1/2)c_2} + a_5[\mathrm{e}^{(1/2)c_2} + c_1c_3\,\mathrm{e}^{-(1/2)c_2}] \tag{2.22}$$

$$\zeta_6 \equiv a_6. \tag{2.23}$$

In the above expression, a dot appearing above each symbol denotes a time derivative. Recalling our requirement, the expression (2.16) is taken to be independent of $\hat{J}_+$, $\hat{J}_-$ and $\hat{J}_0$. In view of (2.17), the above requirement can be recast in the following equations:

$$\zeta_1 = \zeta_2 = \zeta_3 = 0. \tag{2.24}$$

These equations can be rewritten, after some manipulation, into the following:

$$\dot{c}_1 = a_1' + a_2'c_1 - a_3'c_1^2 \tag{2.25}$$

$$\dot{c}_2 = a_2' - 2a_3'c_1 \tag{2.26}$$

$$\dot{c}_3 = a_3'\,\mathrm{e}^{c_2} \tag{2.27}$$

in which we have denoted

$$a_j' \equiv \frac{a_j}{\mathrm{i}\hbar} \qquad j = 1, 2, 3. \tag{2.28}$$

The above three familiar equations are identical to the equations obtained in the previous paper. The first equation (2.25) is the well known Ricatti equation. This can be solved readily as we have done in the previous paper once the coefficients a_j are given. The other two equations (2.27) and (2.28) can hence be solved readily:

$$c_2 = \int_0^t \mathrm{d}u[a_2' - 2a_3'c_1] \tag{2.29}$$

$$c_3 = \int_0^t \mathrm{d}u\, a_3'\,\mathrm{e}^{c_2}. \tag{2.30}$$

Up to this point, the unitary operator represented in (2.12) is completely determined. Now we return to look at the evolution equation (2.15). In view of (2.17) and (2.24), this evolution equation can be rewritten in the following form:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{\mathscr{H}}(t)|\Psi(t)\rangle \tag{2.31}$$

in which

$$\hat{\mathscr{H}}(t) = \zeta_4\hat{K}_+ + \zeta_5\hat{K}_- + \zeta_6. \tag{2.32}$$

We readily realise that the Hamiltonian $\hat{\mathscr{H}}(t)$ consists only of h(4) Lie group generators. This Heisenberg-Weyl group, being simpler than the SU(2) group, allows us to solve the evolution equation (2.31) easily.

By means of the result given in (2.2), the evolution operator for the above equation (2.31) can be written in the following form:

$$\hat{U}_2(t) = e^{c_6} e^{c_4 \hat{K}_+} e^{c_5 \hat{K}_-}. \tag{2.33}$$

This evolution operator satisfies the expression

$$|\Psi(t)\rangle = \hat{U}_2(t)|\Psi(0)\rangle \tag{2.34}$$

and obeys the initial condition

$$\hat{U}_2(0) = \hat{I}. \tag{2.35}$$

This initial condition implies the following initial conditions for $c_j(t)$:

$$c_j(0) = 0 \qquad j = 4, 5, 6. \tag{2.36}$$

Inserting expression (2.33) for the evolution operator back into the evolution equation (2.31), it is not difficult to obtain the following sets of equations for $c_j(t)$ $(j = 4, 5, 6)$:

$$\dot{c}_4 = \zeta'_4 \tag{2.37}$$

$$\dot{c}_5 = \zeta'_5 \tag{2.38}$$

$$\dot{c}_6 = \zeta'_5 c_4 + \zeta'_6 \tag{2.39}$$

where

$$\zeta'_j \equiv \frac{\zeta_j}{\mathrm{i}\hbar} \qquad j = 4, 5, 6. \tag{2.40}$$

The above set of equations can be solved easily to give

$$c_4 = \int_0^t \mathrm{d}u\, \zeta'_4 \tag{2.41}$$

$$c_5 = \int_0^t \mathrm{d}u\, \zeta'_5 \tag{2.42}$$

$$c_6 = \int_0^t \mathrm{d}u [\zeta'_5 c_4 + \zeta'_6]. \tag{2.43}$$

Up to this point, the Schrödinger wavefunction has been completely determined. In view of (2.11) and (2.34), the wavefunction is given by

$$|\Phi(t)\rangle = \hat{U}(t, 0)|\Phi(0)\rangle \tag{2.44}$$

in which $\hat{U}(t, 0)$ is expressible as

$$\begin{aligned} \hat{U}(t, 0) &= \hat{U}_1(t)\hat{U}_2(t) \\ &= e^{c_6} e^{c_1 \hat{J}_+} e^{c_2 \hat{J}_0} e^{c_3 \hat{J}_-} e^{c_4 \hat{K}_+} e^{c_5 \hat{K}_-} \end{aligned} \tag{2.45}$$

where the $c_j(t)$ are solutions to the set of ordinary differential equations in (2.25)-(2.27) and (2.37)-(2.39). Therefore the evolution equation (2.10) is basically solved.

3. Mass-varying harmonic oscillator under an external force

In the last section, we presented a way to evaluate the evolution operator for a SU(2)⊕h(4) Hamiltonian. We are now in a position to apply this method to the problem of a mass-varying harmonic oscillator under an external force. To begin, we write down the Hamiltonian as follows [15-17]:

$$\hat{H}(t)=\frac{\hat{p}^2}{2M(t)}+\tfrac{1}{2}M(t)\omega^2\hat{q}^2+\hat{V}_{\text{ext}} \tag{3.1}$$

where $\hat{V}_{\text{ext}}$ is an external potential arising from an external force

$$\hat{V}_{\text{ext}}=-F(t)\hat{q}. \tag{3.2}$$

The equations of motion for the operators $\hat{p}$ and $\hat{q}$ are found to be

$$\dot{q}=\frac{1}{\mathrm{i}\hbar}[q,H]=\frac{p}{M(t)} \tag{3.3}$$

$$\dot{p}=\frac{1}{\mathrm{i}\hbar}[p,H]=-M(t)\omega^2q+F(t). \tag{3.4}$$

These two equations will give rise to the following equation:

$$\ddot{q}+\frac{\mathrm{d}}{\mathrm{d}t}\ln[M(t)]\dot{q}+\omega^2q=\frac{F(t)}{M(t)} \tag{3.5}$$

which is identical to the classical equation of motion for a harmonic oscillator with damping coefficient $(\mathrm{d}/\mathrm{d}t)\ln[M(t)]$ and under an external force $F(t)$.

To apply the result of the last section, we rewrite the Hamiltonian (3.1) in the form

$$\hat{H}(t)=a_1(t)\hat{J}_++a_2(t)\hat{J}_0+a_3(t)\hat{J}_-+a_4(t)\hat{K}_++a_5(t)\hat{K}_-+a_6(t) \tag{3.6}$$

in which we have denoted

$$a_1(t)=\hbar M(t)\omega^2 \tag{3.7}$$

$$a_2(t)=0 \tag{3.8}$$

$$a_3(t)=\frac{\hbar}{M(t)} \tag{3.9}$$

$$a_4(t)=-\sqrt{\hbar}F(t) \tag{3.10}$$

$$a_5(t)=0 \tag{3.11}$$

$$a_6(t)=0 \tag{3.12}$$

and the operators have the following identification:

$$\hat{J}_+=\frac{1}{2\hbar}\hat{q}^2 \tag{3.13}$$

$$\hat{J}_0=\frac{\mathrm{i}}{4\hbar}(\hat{p}\hat{q}+\hat{q}\hat{p}) \tag{3.14}$$

$$\hat{J}_-=\frac{1}{2\hbar}\hat{p}^2 \tag{3.15}$$

$$\hat{K}_+=\frac{1}{\sqrt{\hbar}}\hat{q} \tag{3.16}$$

$$\hat{K}_-=\frac{\mathrm{i}}{\sqrt{\hbar}}\hat{p}. \tag{3.17}$$

Now according to (2.45), we can write the evolution operator for the above Hamiltonian (3.1) as follows:

$$\hat{U}(t,0)=\mathrm{e}^{c_6}\,\mathrm{e}^{c_1\hat{J}_+}\,\mathrm{e}^{c_2\hat{J}_0}\,\mathrm{e}^{c_3\hat{J}_-}\,\mathrm{e}^{c_4\hat{K}_+}\,\mathrm{e}^{c_5\hat{K}_-} \tag{3.18}$$

in which $c_j = c_j(t), j = 1, \ldots, 6$, satisfy the ordinary differential equations (2.25)-(2.27) and (2.37)-(2.39). Inserting equations (3.7)-(3.12) into this set of equations, we readily obtain

$$\dot{c}_1 = -\mathrm{i}\left[M(t)\omega^2 - \frac{1}{M(t)}c_1^2\right] \tag{3.19}$$

$$\dot{c}_2 = \frac{2\mathrm{i}}{M(t)}c_1 \tag{3.20}$$

$$\dot{c}_3 = -\frac{\mathrm{i}}{M(t)}\mathrm{e}^{c_2} \tag{3.21}$$

$$\dot{c}_4 = \frac{\mathrm{i}}{\sqrt{\hbar}}F(t)\,\mathrm{e}^{-(1/2)c_2} \tag{3.22}$$

$$\dot{c}_5 = \frac{\mathrm{i}}{\sqrt{\hbar}}F(t)c_3\,\mathrm{e}^{-(1/2)c_2} \tag{3.23}$$

$$\dot{c}_6 = \frac{\mathrm{i}}{\sqrt{\hbar}}F(t)c_3c_4\,\mathrm{e}^{-(1/2)c_2} \tag{3.24}$$

with initial condition

$$c_j(0) = 0 \qquad j = 1, \ldots, 6. \tag{3.25}$$

Once $M(t)$ is specified, the set of differential equations (3.19-(3.24) can be readily solved.

As we have done in the previous paper, we assume an exponential variation for the mass of the oscillator

$$M(t) = m\,\mathrm{e}^{\gamma t}. \tag{3.26}$$

The damping parameter γ is here taken to be at the 'critically damped' value [1]:

$$\gamma = 2\omega. \tag{3.27}$$

On the other hand, the external force is taken to be sinusoidal with frequency ω_1

$$F(t) = f\cos(\omega_1 t + \theta) \tag{3.28}$$

where f denotes the amplitude of the force and θ signifies a phase difference between the external force and the oscillator. Having specified the mass and the external force, we can readily solve the above set of differential equations. The result is given below:

$$c_1 = -\mathrm{i}m\omega^2\frac{t\,\mathrm{e}^{2\omega t}}{1+\omega t} \tag{3.29}$$

$$c_2 = 2\omega t - 2\ln[1+\omega t] \tag{3.30}$$

$$c_3 = -\frac{\mathrm{i}}{m}\frac{t}{1+\omega t} \tag{3.31}$$

$$c_4 = \mathrm{i}\sqrt{m\omega}\,\tilde{f}\frac{1}{1+\tilde{\omega}_1^2}[\mathrm{e}^{-\omega t}h(t) - h(0)] \tag{3.32}$$

$$c_5 = \frac{\tilde{f}}{\sqrt{m\omega}}\frac{1}{1+\tilde{\omega}_1^2}[\mathrm{e}^{-\omega t}g(t) - g(0)] \tag{3.33}$$

$$c_6 = \mathrm{i}\theta(t) \tag{3.34}$$

in which we have denoted

$$\tilde{f}=\frac{f}{\sqrt{\hbar m\omega^2}} \tag{3.35}$$

$$\tilde{\omega}=\frac{\omega_1}{\omega} \tag{3.36}$$

$$g(t)=\left(-\omega t+\frac{\tilde{\omega}^2-1}{\tilde{\omega}^2+1}\right)\cos(\omega_1 t+\theta)+\left(\omega_1 t+\frac{2\tilde{\omega}}{\tilde{\omega}^2+1}\right)\sin(\omega_1 t+\theta) \tag{3.37}$$

$$h(t)=-\left(\omega t+\frac{2}{\tilde{\omega}^2+1}\right)\cos(\omega_1 t+\theta)+\tilde{\omega}\left(\omega_1 t+\frac{\tilde{\omega}+3}{\tilde{\omega}^2+1}\right)\sin(\omega_1 t+\theta) \tag{3.38}$$

and $\theta(t)$ is a real function of time.

In the above expression for $c_6(t)$, we have not given the explicit expression for $\theta(t)$. This is because $c_6(t)$ is only a phase factor which will not enter into our later calculation.

We have here found the evolution operator of the oscillator. Hence we can proceed to evaluate the expectation value for the self-energy of the oscillator:

$$\begin{aligned} E(t)&=\langle\Phi(t)|\hat{H}_0(t)|\Phi(t)\rangle \\ &=\langle\alpha|\hat{U}^{+}(t,0)\hat{H}_0(t)\hat{U}(t,0)|\alpha\rangle. \end{aligned} \tag{3.39}$$

In the above, $\hat{H}_0(t)$ denotes the self-energy part of the time-dependent Hamiltonian (3.1)

$$\hat{H}_0(t)=\frac{\hat{p}^2}{2M(t)}+\tfrac{1}{2}M(t)\omega^2\hat{q}^2. \tag{3.40}$$

Also, we begin with an initial coherent state $|\Phi(0)\rangle=|\alpha\rangle$ with

$$\alpha=\sqrt{n_0}\,e^{i\varphi}. \tag{3.41}$$

After some algebra, we arrive at

$$E(t)=E_0(t)+E_f(t) \tag{3.42}$$

in which $E_0(t)$ is the energy expectation value for a mass-varying harmonic oscillator with no external force, which is identical to the result in the previous paper:

$$E_0(t)=\hbar\omega\{(n_0+\tfrac{1}{2})(1+2\omega^2t^2)+2n_0\omega t\cos 2\varphi+2n_0\omega^2t^2\sin 2\varphi\} \tag{3.43}$$

and $E_f(t)$ corresponds to the part of energy due to the presence of the external force $F(t)$,

$$E_f(t)=\hbar\omega\{\sqrt{2n_0}\,\tilde{f}(\mathscr{E}_1\cos\varphi-\mathscr{E}_2\sin\varphi)+\tilde{f}^2\mathscr{E}_3\} \tag{3.44}$$

in which

$$\mathscr{E}_1=-\frac{1}{\sqrt{\hbar}f}\,e^{-(1/2)c_2}[a_1c_5+a_3c_1c_4] \tag{3.45}$$

$$\mathscr{E}_2=i\frac{m\omega}{\sqrt{\hbar}f}\,e^{-(1/2)c_2}[a_1c_3c_5+a_3c_4(c_1c_3+e^{c_2})] \tag{3.46}$$

$$\mathscr{E}_3=\frac{m\omega^2}{2f^2}\,[a_1c_5^2-a_3c_4^2]. \tag{3.47}$$

In the same way, we can evaluate the expectation values for the position $\bar{q}$ and the momentum $\bar{p}$ of the oscillator:

$$\bar{q} = \bar{q}_0 + \bar{q}_f \tag{3.48}$$

$$\bar{p} = \bar{p}_0 + \bar{p}_f \tag{3.49}$$

where

$$\bar{q}_0 = \sqrt{\frac{2\hbar n_0}{m\omega}}\, e^{-\omega t}\{(1+\omega t)\cos\varphi + \omega t \sin\varphi\} \tag{3.50}$$

$$\bar{q}_f = -\sqrt{\hbar}\, c_5 \tag{3.51}$$

$$\bar{p}_0 = \sqrt{2\hbar m\omega n_0}\, e^{\omega t}\{(1-\omega t)\sin\varphi - \omega t\cos\varphi\} \tag{3.52}$$

$$\bar{p}_f = -i\sqrt{\hbar}\, c_4. \tag{3.53}$$

It could be easily identified that $\bar{q}_0$ and $\bar{p}_0$ correspond to the expectation values for a mass-varying harmonic oscillator with no external force; while $\bar{q}_f$ and $\bar{p}_f$ arise from the presence of the external force.

In order to discuss the squeezing property of the oscillator, we need to calculate the variance of the position and the momentum of the oscillator. They are listed below:

$$\Delta q^2 = \Delta q_0^2 = \frac{\hbar}{2m\omega}\, e^{-2\omega t}\{1 + 2\omega t + 2\omega^2 t^2\} \tag{3.54}$$

$$\Delta p^2 = \Delta p_0^2 = \frac{\hbar m\omega}{2}\, e^{2\omega t}\{1 - 2\omega t + 2\omega^2 t^2\}. \tag{3.55}$$

The result here shows that the variances of the two quantities are identical to their counterpart for the mass-varying harmonic oscillator with no external force. In other words, the two variances are not changed upon the introduction of the external force. From the result of the previous paper, we know that there is squeezing in the fluctuation in the position of the oscillator. This indicates that the initial wavefunction in (3.41) will evolve as a squeezed state.

From equations (3.54) and (3.55), the uncertainty relation is given by

$$\Delta p \Delta q = \frac{\hbar}{2}(1 + 4\omega^4 t^4)^{1/2} \tag{3.56}$$

which simply means that the product uncertainty will not reach the minimum value of $\hbar/2$ as time proceeds.

Now we introduce a set of operators as follows:

$$\hat{A} = \hat{U}(t,0)\hat{a}\,\hat{U}^+(t,0) \tag{3.57}$$

$$\hat{A}^+ = \hat{U}(t,0)\hat{a}^+\hat{U}^+(t,0) \tag{3.58}$$

where $\hat{a}$, $\hat{a}^+$ are the usual annihilation and creation operators for a harmonic oscillator. The new set of operators satisfy the same commutation relation as $\hat{a}$ and $\hat{a}^+$:

$$[\hat{A}, \hat{A}^+] = 1. \tag{3.59}$$

Since we begin with a coherent state

$$|\Phi(0)\rangle = |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,\hat{a}^{+n}|0\rangle \tag{3.60}$$

it can be easily shown that the wavefunction for the mass-varying harmonic oscillator $|\Phi(t)\rangle$ is a coherent state with respect to the new set of operators:

$$\hat{A}|\Phi(t)\rangle = \alpha|\Phi(t)\rangle. \tag{3.61}$$

With the help of (3.18) and

$$\hat{a}^{+} = \frac{1}{\sqrt{2m\hbar\omega}}[m\omega\hat{q} - \mathrm{i}\hat{p}] \tag{3.62}$$

we obtain the transformation from the ordinary set of operators $\hat{a}$, $\hat{a}^{+}$ to the new set, $\hat{A}$, $\hat{A}^{+}$:

$$\hat{A} = \eta_1^*(t)\hat{a} - \eta_2(t)\hat{a} - \frac{1}{\sqrt{2m\omega}}c_4(t) + \sqrt{\frac{m\omega}{2}}c_5(t) \tag{3.63}$$

in which

$$\eta_1 = \tfrac{1}{2}\,\mathrm{e}^{-(1/2)c_2}\left[1 + c_1c_3 + \mathrm{e}^{c_2} + \frac{1}{m\omega}c_1 + m\omega c_3\right] \tag{3.64}$$

$$\eta_2 = \tfrac{1}{2}\,\mathrm{e}^{-(1/2)c_2}\left[1 - c_1c_3 - \mathrm{e}^{c_2} + \frac{1}{m\omega}c_1 - m\omega c_3\right] \tag{3.65}$$

and they are found to satisfy

$$|\eta_1|^2 - |\eta_2|^2 = 1. \tag{3.66}$$

Since the transformation (3.63) here is not a Bogoliubov-type transformation, we can conclude that the wavefunction $|\Phi(t)\rangle$ does not evolve as a squeezed coherent state. (In the previous paper, the wavefunction for a mass-varying harmonic oscillator with no external force is a squeezed coherent state.) Instead, the wavefunction $|\Phi(t)\rangle$ for our present problem is simply a squeezed state.

4. Remarks

In this paper, we have devised a method to find the evolution operator for a Hamiltonian underlying a SU(2)$\oplus$h(4) group structure. We have not directly employed the Magnus expansion method studied in the previous paper. Instead, we invoke a transformation to the original wavefunction so that the transformed Hamiltonian will take a simpler group structure. The group theoretic method presented in the previous paper can then be employed to obtain the whole evolution operator.

We have studied the problem of a mass-varying harmonic oscillator under an external force. It is found that in general the expectation values for the energy, the position and the momentum of the oscillator with an external force are different from those with no external force. However the fluctuations Δp^2, Δq^2 and $\Delta p\Delta q$ remain the same whatever the external force.

In conclusion, we would like to remark that the group theoretic method is very useful in obtaining an evolution operator for a Hamiltonian underlying a closed Lie group structure. This group theoretic method allows us to obtain an exact form for an evolution operator. In the case of a Hamiltonian possessing a more complex group structure, the transformation method we invoke in section 2 will become useful in simplifying the group structure. Therefore the combination of the group theoretic method and the transformation method allows us to treat a broader class of Hamiltonian systems underlying a closed Lie group structure.

References

[1] Cheng C M and Fung P C W 1988 *J. Phys. A: Math. Gen.* **21** 4115-31
[2] Magnus W 1954 *Commun. Pure Appl. Math.* **7** 649-73
[3] Wei J and Norman E 1963 *J. Math. Phys.* **4** 575-81
[4] Wódkiewicz W and Eberly J H 1985 *J. Opt. Soc. Am.* B **2** 458-66
[5] Kanai E 1948 *Prog. Theor. Phys.* **3** 440-2
[6] Colegrave R K and Abdalla M S 1981 *J. Phys. A: Math. Gen.* **14** 2269-80
[7] Colegrave R K and Abdalla M S 1982 *J. Phys. A: Math. Gen.* **15** 1549-59
[8] Leach P G L 1983 *J. Phys. A: Math. Gen.* **16** 3261-69
[9] Abdalla M S and Colegrave R K 1984 *Lett. Nuovo Cimento* **39** 373-9
[10] Abdalla M S 1985 *Lett. Nuovo Cimento* **44** 482-6
[11] Abdalla M S 1986 *Phys. Rev.* A **33** 2870-76
[12] Cheng B K 1985 *Phys. Lett.* **113A** 293-6
[13] Fernández F M 1987 *J. Math. Phys.* **28** 2908
[14] Ben-Aryeh Y and Mann A 1985 *Phys. Rev. Lett.* **54** 1020-2
[15] Tartaglia A 1977 *Lett. Nuovo Cimento* **19** 205-9
[16] Dodonov V V and Man'ko V I 1979 *Phys. Rev.* A **20** 550-60
[17] Khandekan D C and Lawende S V 1979 *J. Math. Phys.* **20** 1870-7
[18] Dattoli G, Richetta M and Torre A 1988 *Phys. Rev.* A **37** 2007-11